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UCRL-JC-116094
PREPRINT

TENSOR ARTIFICIAL VISCOSITY AND ARTIFICIAL HEAT CONDUCTION IN SPHERICAL GENERAL RELATIVISTIC COLLAPSE

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This paper was prepared for submittal to
General Relativity and Gravitation

January 1994

Lawrence Livermore National Laboratory

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Suggested Running Head: Tensor Artificial Viscosity in General Relativistic Collapse

Received______________________________

Interesting problems in gravitational collapse often involve modeling shock phenomena. Typically, an artificial viscosity method is used to accomplish this. Here we formulate equations of spherically symmetric general relativistic hydrodynamics which include more sophisticated forms of the artificial viscosity and also include an artificial heat flux.

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1: INTRODUCTION

Many problems in spherical general relativistic hydrodynamics involve modeling shocks. In particular, modeling supernova core collapse and the subsequent explosion require shocks to be evolved accurately. In order to do this, the artificial viscosity method has frequently been used [1]. This method is used in non-relativistic applications very successfully, and it is employed in relativistic calculations in essentially the same manner: a term is added to the pressure of the system which is significant only when the velocity is changing rapidly. This term is referred to as a scalar artificial viscosity, and mimics the effects of the real bulk viscosity of the material [2]. It allows for the generation of entropy by a shock. Here we describe a modification of this prescription. We add terms to the stress-energy tensor of the material which mimic the contributions of both bulk and shear viscosity. These are referred to as a tensor artificial viscosity. Other terms mimic radial heat flow, and are referred to as an artificial heat conduction. The use of these frequently allows for more accurate modeling of shocks, particularly in spherical problems [3].

We present the development of spherical general relativistic hydrodynamics including these terms. A subsequent paper will discuss the results of stellar collapse calculations using these equations.

2. DEVELOPMENT OF THE EQUATIONS

The physical constants G and c will be set equal to one in the following. We start with a spherically symmetric metric in comoving coordinates [1]:

$$ds^2 = a^2dt^2 - b^2d\mu^2 - R^2d\Omega^2 \tag{1}$$

Here a is a temporal metric coefficient, and b is a radial one. We use the rest mass μ as our radial coordinate. R is the areal radius. The quantity $\phi \equiv \ln(a)$ is sometimes used in place of the quantity a. We wish to solve the Einstein equations with this metric and a stress-energy tensor that respects spherical symmetry. The most general form for this tensor is [4]

$$T_{\hat{a}\hat{b}} = \begin{bmatrix} \rho(1+\varepsilon) & H & & \\ H & P+Q_R & & \\ & & P+Q_A & \\ & & & P+Q_A \end{bmatrix}. \tag{2}$$

Here ρ is the locally observed rest mass density, ε the locally observed energy per unit mass, and P the locally observed pressure. In scalar artificial viscosity formulations, the is no off-diagonal term H, and $Q_R = Q_A$; the pressure plus the viscous pressure is isotropic. Here, we modify the stress energy tensor in two ways. We allow for heat flow in the radial direction, and we no longer force the viscous pressure to be isotropic, although it is the same in the two angular directions. Q_R is an artificial viscosity term which acts like a pressure in the radial direction, and Q_A is an analogous term which acts in the angular directions perpendicular to the radial one. H acts like a heat flow in the radial direction. The components of this tensor are written in the local inertial frame, as denoted by the carets over the subscripts. Note that this is different than the comoving coordinate frame.

Given the metric and the stress-energy tensor, we can write down the Einstein equations $G_{\hat{a}\hat{b}} = 8\pi T_{\hat{a}\hat{b}}$. Components of the Einstein tensor can be found in [4]. The following draws heavily on [1] and [5]. The addition of our new terms makes the development, and the resultant hydrodynamics calculations, more complex.

There are four non-trivial components of $G_{\hat{a}\hat{b}}$:

$$G_{\hat{t}}^{\hat{t}} = -2\left[\frac{\dot{b}\dot{R}}{a^2 bR} - \left(\frac{R'}{b}\right)'\frac{1}{bR}\right] - \frac{1}{R^2} - \frac{\dot{R}^2}{a^2R^2} + \frac{R'^2}{b^2R^2} \tag{3}$$

$$G_{\hat{\mu}}^{\hat{t}} = \frac{2}{abR}\left[\dot{R}' - \dot{R}\frac{a'}{a} - R'\frac{\dot{b}}{b}\right] \tag{4}$$

$$G_{\hat{\mu}}^{\hat{\mu}} = -2\left[\left(\frac{\dot{R}}{a}\right)^{\cdot}\frac{1}{aR} - \frac{a'R'}{ab^2R}\right] - \frac{1}{R^2} - \frac{\dot{R}^2}{a^2R^2} + \frac{R'^2}{b^2R^2} \tag{5}$$

$$\begin{aligned} G_{\hat{\theta}}^{\hat{\theta}} = G_{\hat{\varphi}}^{\hat{\varphi}} = -\Bigg[& \frac{\dot{b}\dot{R}}{a^2bR} - \left(\frac{R'}{b}\right)'\frac{1}{bR} \\ & + \left(\frac{\dot{R}}{a}\right)^{\cdot}\frac{1}{aR} - \frac{a'R'}{ab^2R} \\ & + \left\{\left(\frac{\dot{b}}{a}\right)^{\cdot} - \left(\frac{a'}{b}\right)'\right\}\frac{1}{ab}\Bigg] \end{aligned} \tag{6}$$

Here $\bullet \equiv \partial_t$ and $' \equiv \partial_\mu$.are derivatives with respect to t and μ.

The $^{\hat{t}}_{\hat{t}}$, $^{\hat{t}}_{\hat{\mu}}$, and $^{\hat{\mu}}_{\hat{\mu}}$ Einstein equations become, after some algebra, respectively:

$$4\pi[\rho(1+\varepsilon)]R^2R' + 4\pi R^2\dot{R}\frac{b}{a}H = \frac{1}{2}\left\{R - \frac{RR'^2}{b^2} + \frac{R\dot{R}^2}{a^2}\right\}', \quad (7)$$

$$4\pi abRH = \dot{R}' - \dot{R}\frac{a'}{a} - R'\frac{\dot{b}}{b}, \quad (8)$$

and

$$4\pi[P + Q_R]R^2\dot{R} + 4\pi R^2R'\frac{a}{b}H = -\frac{1}{2}\left\{R - \frac{RR'^2}{b^2} + \frac{R\dot{R}^2}{a^2}\right\}^{\cdot}. \quad (9)$$

We also have two equations from stress-energy conservation, $\nabla_a T^{ab} = 0$. Because of the Bianchi identity, $\nabla_a G^{ab} = 0$,we can replace two of the equations that result from $G_{\hat{a}\hat{b}} = 8\pi T_{\hat{a}\hat{b}}$. We choose to replace the $^{\hat{\theta}}_{\hat{\theta}}$ and $^{\hat{\varphi}}_{\hat{\varphi}}$ equations. Stress-energy conservation can be written in the comoving coordinate basis as [6]

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^a}\left(\sqrt{-g}\,T^{ab}\right) + \Gamma^b_{cd}T^{cd} = 0. \quad (10)$$

The affine connections Γ^a_{bc} can be found from Ref. 4, pp. 248-251 ; $g = -a^2b^2R^4\sin^2\theta$ denotes the determinant of the metric tensor. There are two non-trivial equations that result from $\nabla_a T^{ab} = 0$, corresponding to the t and μ derivatives in equation (10):

$$\begin{aligned}&[\rho(1+\varepsilon)]^{\cdot} + [\rho(1+\varepsilon) + P]\left(\frac{\dot{b}}{b} + \frac{2\dot{R}}{R}\right)\\ &+\frac{a}{b}\left[H' + \frac{2R'}{R}H\right] + \frac{2a'}{b}H\\ &+\frac{\dot{b}}{b}Q_R + \frac{2\dot{R}}{R}Q_A = 0\end{aligned} \quad (11)$$

and

$$
\begin{aligned}
&\frac{a'}{a}[\rho(1+\varepsilon)] + P' + Q_R' \\
&+\frac{b}{a}\left[\dot{H} + \frac{2\dot{R}}{R}H\right] + \frac{2\dot{b}}{a}H \\
&+Q_R\left(\frac{a'}{a} + \frac{2R'}{R}\right) - Q_A\frac{2R'}{R} = 0.
\end{aligned}
\tag{12}
$$

We assume that rest mass is conserved. This provides an additional equation. This is a constraint of the density and fluid four-velocity $u^a = \frac{dx^a}{d\tau}$. We require $\nabla_a(\rho u^a) = 0$. This covariant divergence can be written in a coordinate basis [6] as:

$$
\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^a}\left(\sqrt{-g}\,\rho u^a\right) = 0,
\tag{13}
$$

and in a coordinate basis $u^a = (\frac{1}{a},0,0,0)$, so rest mass conservation becomes

$$
\frac{\partial}{\partial t}\left(\rho b R^2\right) = 0.
\tag{14}
$$

Given an equation of state $P(\rho,\varepsilon)$ and functional forms for Q_R, Q_A, and, H, our equations form a well-defined hyperbolic set. We now manipulate the equations to develop a set of equations that resemble the usual spherical general relativistic equations.

First, we eliminate the metric coefficient b using equation (14). We define the metric coefficient $b = 1/(4\pi\rho R^2)$. This is the usual choice and ensures that the radial coordinate is rest mass [2],[5].

Using this expression, we can eliminate b in equation (11) to obtain an expression for the time derivative of the energy density:

$$
\dot{\varepsilon} = -(P + Q_R)\left[\frac{1}{\rho}\right]^{\cdot} + \frac{2au}{\rho R}(Q_R - Q_A) - \frac{4\pi}{a}\left[a^2R^2H\right]'
\tag{15}
$$

Here we introduce the proper velocity

$$
u = \frac{\dot{R}}{a}.
\tag{16}
$$

We define m, the gravitational mass:

$$m = \int_0^{\mu} 4\pi R^2 \rho(1+\varepsilon)\frac{dR}{d\mu} + \frac{uH}{\rho}\ d\mu \tag{17}$$

Integrating equation (7) with respect to μ yields

$$m = \frac{1}{2}\left[R + \frac{R\dot{R}^2}{a^2} - \frac{RR'^2}{b^2}\right]. \tag{18}$$

We define

$$\Gamma = \frac{R'}{b} = 4\pi\rho R^2\frac{dR}{d\mu}, \tag{19}$$

the proper volume coefficient. Solving equation (18) for Γ yields

$$\Gamma = \left[1 + u^2 - \frac{2m}{R}\right]^{\frac{1}{2}}. \tag{20}$$

We now obtain an expression for the spatial derivative of the temporal metric coefficient a. We solve equation (12) for $\frac{a'}{a}$:

$$\frac{a'}{a} = -\frac{\left(P + Q_R\right)'}{\left(\rho(1+\varepsilon) + P + Q_R\right)} - \frac{C}{\left(\rho(1+\varepsilon) + P + Q_R\right)}, \tag{21}$$

where

$$C = \left(Q_R - Q_A\right)\frac{2\Gamma}{4\pi\rho R^3} + \frac{\rho}{4\pi a}\left[\frac{H}{\rho^2 R^2}\right]^{\cdot}. \tag{22}$$

In terms of ϕ, we have

$$\phi' = -\frac{\left(P + Q_R\right)'}{\left(\rho(1+\varepsilon) + P + Q_R\right)} - \frac{C}{\left(\rho(1+\varepsilon) + P + Q_R\right)}. \tag{23}$$

Equation (22) or (24) is integrated inward from the edge of the star. A boundary condition must be provided. In [2], the choice a = 1 is made; in [5], ϕ at the edge is set equal to the log of the Schwarzschild temporal metric coefficient:

$$\phi_{edge} = \ln\left(\left[1-\frac{2GM}{Rc^2}\right]\left[1+\frac{u^2}{c^2}-\frac{2GM}{Rc^2}\right]^{-1/2}\right) \tag{24}$$

Using equations (16), (18), (21) and (22) along with equations (8) and (9), we obtain an expression for the time derivative of the proper velocity:

$$\begin{aligned}\dot{u} = -a\Bigg[& 4\pi R^2\Gamma\frac{1}{\left(1+\varepsilon+\frac{P+Q_R}{\rho}\right)}(P+Q_R)' + \frac{m}{R^2} \\ & + 4\pi(P+Q_R)R \\ & 4\pi R^2\Gamma\frac{1}{\left(1+\varepsilon+\frac{P+Q_R}{\rho}\right)}C\Bigg]\end{aligned} \tag{25}$$

Eliminating b in equation (8) gives an equation for the time derivative of rest mass density

$$\frac{(\rho R^2)^{\cdot}}{\rho R^2} = -\frac{au'}{R'} + \frac{4\pi bRH}{R'}, \tag{26}$$

which corresponds to equation (32) in [2]. Alternately , as in [5], one can update the rest mass density by using the fact that the rest mass of a zone is constant, i.e.

$$(\Delta\dot{\mu}) = \left[\frac{4\pi\rho R^2\Delta R}{\Gamma}\right]^{\cdot} = 0. \tag{27}$$

Solving this equation for ρ in terms of R and Γ yields the density of the zone spanned by ΔR.

It proves useful to define the enthalpy w, as it appears in many of the equations above:

$$w = 1+\varepsilon+\frac{P+Q_R}{\rho} \tag{28}$$

A full list of the equations with G's and c's restored is provided in Table I.

The equations above conserve both rest mass and energy. This can be shown by explicitly exhibiting equations for these quantities in conservation form. Manipulation of the equations produces:

$$\frac{\partial}{\partial t}\left[\frac{\Gamma}{\rho}\right]-\frac{\partial}{\partial\mu}\left[4\pi R^2 au\right]=0 \tag{29}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\Gamma\left(1+\frac{\varepsilon}{c^2}\right)c^2+\frac{Hu}{\rho c^2}\right]\\&-\frac{\partial}{\partial\mu}\left[4\pi R^2 au\left(P+Q_R\right)+4\pi R^2 aH\Gamma\right]=0\end{aligned} \tag{30}$$

The first equation describes the relationship between proper volume per unit rest mass and the change in coordinate volume. They change in such a way as to keep the rest mass of each zone constant. The second equation shows that if H and Q_R are zero at the origin and on the boundary of the system, the total mass of the system remains constant. These equations correspond to equations (47) and (48) of [2].

3. FORMS FOR Q AND H

Both components of the artificial viscosity, Q_R and Q_A, have the units of pressure, force per area. The artificial heat flux H has the units of an energy flux, energy per unit area per time. Different forms for Q and H perform differently in modeling situations such as wall heating, shocks passing through density gradients, shockless compression, and unequal zone sizes in simulations. Choosing the best implementation of these quantities for a particular problem requires some experimentation. We will examine some general considerations below. A detailed study of the effects of different formulations for Q and H is being undertaken.

First we will examine artificial viscosities alone, i.e. H = 0. The simplest case is the scalar artificial viscosity. This obtains when $Q_R = Q_A$. Then C = 0, and we recover the familiar equations of [2] and [5]. The artificial viscosity then shows up everywhere in the equations as a single function Q added to the pressure P. Many different functional forms for artificial viscosities have been proposed. Refs. 1 and 3 discuss several. These typically depend on velocity gradients; some modifications are functions of rates of change of density, sound speed, or more sophisticated functions of the velocity. A widely used form is that given in von Neumann and Richtmeyer [7]:

$$Q_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \begin{cases} 2\rho\left(\Delta u^2\right)_{j+\frac{1}{2}}^{n+\frac{1}{2}} & \text{if } \Delta u_{j+\frac{1}{2}}^{n+\frac{1}{2}} = u_{j+1}^{n+\frac{1}{2}} - u_{j}^{n+\frac{1}{2}} < 0 \\ 0 & \text{if } \Delta u_{j+\frac{1}{2}}^{n+\frac{1}{2}} > 0 \end{cases} . \tag{31}$$

In non-relativistic calculations, u is the coordinate velocity. In relativistic situations, the proper velocity is used.

A tensor artificial viscosity requires $Q_R \neq Q_A$. In this case, we have two separate functions Q_R and Q_A, for which to give a functional form. Typically, however, Q_A is chosen to be some multiple of Q_R, which is calculated as a scalar Q is. The relativistic equations can be reduced to non-relativistic equations by taking $G \to 0$, $\frac{1}{c^2} \to 0$. The non-relativistic equation for the time derivative of velocity is

$$\dot{u} = -4\pi R^2 \frac{\partial(P+Q_R)}{\partial m} - \frac{Q_R - Q_A}{\rho R}. \tag{32}$$

The corresponding equation, number (1) from [8] is

$$\dot{u} = -\frac{1}{\rho}\frac{\partial(P+Q)}{\partial R} - \frac{3Q}{\rho R}. \tag{33}$$

Comparison of the first term of these equations shows that $Q_R = Q$. Comparing the second term then shows that $Q_A = -\frac{1}{2}Q_R$. A similar comparison to the hydrodynamic equations of Schulz [9], shows that he has implemented the choice. $Q_A = 0$. Whalen [10] has chosen. $Q_A = -\frac{1}{2}Q_R$. Different functions have been chosen for Q_R in all three cases, however.

Whalen [10] gives

$$Q_R = (3/2)(C_0 l)^2 \rho \frac{\partial v}{\partial R}\left(\frac{\partial v}{\partial R} - \frac{1}{3}\nabla \bullet \bar{v}\right); \tag{34}$$

Janke [8] gives

$$Q_R = 3l^2\rho\frac{\partial(R^2v)}{\partial R^3}\left(\frac{\partial v}{\partial R} - \frac{\partial(R^2v)}{\partial R^3}\right), \tag{35}$$

where C_0 and l are constants, v is coordinate velocity and $\vec{v}$ is the coordinate velocity viewed as a three-vector.

There are several ways of constructing a difference approximation to this Q_R, depending on whether the partial derivatives are expanded. Whalen expands them and sets l = ΔR, yielding

$$Q_R = \begin{cases} C_0^2\rho\Delta u\left(\Delta u - \frac{u\Delta R}{R}\right) & \text{if } \Delta u < 0 \\ 0 & \text{if } \Delta u > 0 \end{cases} \tag{36}$$

Janka, following Tscharnuter and Winkler [11], differences Q_R with the derivatives intact:

$$Q_R = \begin{cases} C_0^2\rho\frac{\Delta(R^2u)}{\Delta(R^3)}\left(\frac{\Delta u}{R} - \frac{\Delta(R^2u)}{\Delta(R^3)}\right) & \text{if } \frac{\Delta(R^2u)}{\Delta(R^3)} < 0 \\ 0 & \text{if } \frac{\Delta(R^2u)}{\Delta(R^3)} > 0 \end{cases} \tag{37}$$

The artificial heat conduction H has been studied extensively by Noh [3]. It is a function of gradients in velocity and in thermal energy, and is non-zero only when the artificial viscosity is large. It is an attempt to mimic the heat transfer that occurs between adjacent elements of a fluid when a shock passes through the material. This allows for shocks to develop more smoothly. This is useful in cases where spurious uneven heating occurs, such as when fluid impacts a wall, or in a spherical collapse. In these cases, a large velocity gradient occurs. Most implementations of artificial viscosities produce too much heat, causing a spike in the energy density and density. The artificial heat conduction also allows for a smaller value of artificial viscosity to be used. This is useful if errors proportional to the value of Q arise, such as when shocks propagate through density gradients. Noh gives his standard artificial heat conduction as

$$
H_j^{n+1} = \begin{cases} 2h_0^2\left[\dfrac{\rho\Delta u_{j+\frac{1}{2}}^{n+\frac{1}{2}} \cdot \rho\Delta u_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{\rho\Delta u_{j+\frac{1}{2}}^{n+\frac{1}{2}} + \rho\Delta u_{j-\frac{1}{2}}^{n+\frac{1}{2}}}\right]\left(\varepsilon_{\text{th}\,j+\frac{1}{2}}^{\;n} - \varepsilon_{\text{th}\,j-\frac{1}{2}}^{\;n}\right) & \text{if } Q_{R\,j+\frac{1}{2}}^{\;n+\frac{1}{2}} > 0 \\ 0 & \text{if } Q_{R\,j+\frac{1}{2}}^{\;n+\frac{1}{2}} = 0 \end{cases} \tag{38}
$$

Here, ε_{th} is the thermal part of the energy density. Only a gradient in thermal energy gives rise to a heat flux. For an ideal gas equation of state, all the energy density is thermal. For a typical nuclear equation of state, for example, most of the energy density is due to internal energy in the zero temperature portion of the EOS. A gradient in this internal energy would result in H = 0.

4. SUGGESTED DIFFERENCE FORMULATION

Here we list a set of difference equations for the above hydrodynamics equations. This difference formulation has been implemented and used to test the effects of various choices for Q and H on several different test problems. These equations are essentially those of Refs. [2] and [5] accept for the addition of the terms in Q and H. The innermost edge, at R = 0, is numbered zero, the outermost edge, the boundary of the star, is numbered J. Some quantities are located in zone centers, others on zone edges. We use the notation of Ref. [2]. Superscripts denote time centering, and subscripts denote spatial centering. Half-integer lower subscripts denote zone centered quantities. Zone centered quantities are ρ, ε, P, and w. Edge centered quantities are R, u, m, μ, H, and C. The rest mass μ is used as the radial coordinate and does not change with time. The differences Δμ and ΔR. are zone centered. a and Γ may be centered in either way, depending on the method by which they are calculated. Both methods are described below. Artificial viscosities come into the equations in a manner similar to that of the pressure, and so are naturally zone centered quantities. The artificial heat conduction is an energy flux and so is more naturally calculated at zone boundaries.

Often, good centering requires that an equation involve the value of a quantity at a place where it is not centered, for example ρ_j^{n+1} or $P_{j-\frac{1}{2}}^{n+\frac{1}{2}}$. In this case, averages are taken.

The following difference equations allow the calculation of time n+1 quantities from time n quantities. The equations are explicit - time n+1 quantities are explicit functions of time n quantities. The only exception is the temporal metric coefficient a, which can be calculated only after most other quantities are updated. One can either accept that a is badly centered, or do two sweeps of the following equations, calculating in the first sweep a provisional time n+1 value used in the second sweep.

Velocity is offset half a step in time:

$$
\begin{aligned}
u_j^{n+½} = u_j^{n-½} - a_j^n \Delta t^n \\
\times \Bigg\{ 4\pi\left(R_j^n\right)^2 \frac{\Gamma_j^n}{w_j^n} \frac{P_{j+½}^n + Q_{R\,j+½}^{\;n} - P_{j-½}^n - Q_{R\,j-½}^{\;n}}{\Delta\mu_j} \\
+ \frac{Gm_j^n}{\left(R_j^n\right)^2} + \frac{4\pi G}{c^2}\left(P_j^n + Q_{R_j}^{\;n}\right)R_j^n \\
+ 4\pi\left(R_j^n\right)^2 \frac{\Gamma_j^n}{w_j^n} C_j^n \Bigg\}
\end{aligned}
\tag{39}
$$

This offset makes the radius update equation second order in time, rather than first:

$$
R_j^{n+1} = R_j^n + \Delta t^{n+½} a_j^{n+½} u_j^{n+½} \tag{40}
$$

Rest mass density can be updated in a way that resembles that of Ref. 2, or in the way suggested by Ref. 5.

$$
\frac{u'}{R'}{}_{j-½}^{\;n+½} = \frac{u_j^{n+½} - u_{j-1}^{n+½}}{R_j^{n+½} - R_{j-1}^{n+½}} \tag{41}
$$

$$
f = a_{j-½}^{n+½} \frac{u'}{R'}{}_{j-½}^{\;n+½} \Delta t^{n+½} \tag{42}
$$

$$
\rho_{j-½}^{n+1} = \rho_{j-½}^{n} \frac{(R^2)_{j-½}^{n}}{(R^2)_{j-½}^{n+1}} \exp(f) \tag{43}
$$

or

$$
\rho_{j-½}^{n+1} = \frac{\Delta\mu_{j-½}}{4\pi} \frac{\Gamma_{j-½}^{n+1}}{\left(R_j^{n+1}\right)^3 - \left(R_{j-1}^{n+1}\right)^3} \tag{44}
$$

With the new density and velocity, we can calculate the artificial viscosity, which we center at time n+1/2. Q_R can be calculated according to many different prescriptions; below, we use the standard Lagrange Q.

$$
Q_{R\,j+½}^{\;n+½} = \begin{cases} 2\rho\left(\Delta u^2\right)_{j+½}^{n+½} & \text{if } \Delta u_{j+½}^{n+½} < 0 \\ 0 & \text{if } \Delta u_{j+½}^{n+½} > 0 \end{cases} \tag{45}
$$

Q_A is set to be some multiple of Q_R:

$$Q_{A\,j+\frac{1}{2}}^{\,n+\frac{1}{2}} = \begin{cases} Q_R & \text{for scalar formulation} \\ -\frac{1}{2}Q_R & \text{for Janka's and Whalen's formulation} \\ 0 & \text{for Shultz's formulation} \end{cases} \tag{46}$$

Energy density can be updated after the new density is calculated. The pressure centered at time n+1/2 can be obtained by calling the EOS with an average of the new density and old density and a guess for the energy density at time n+1/2, or can be formed by an average of old and new pressures. In this later case, we have an implicit equation which must be solved iteratively, since the time n pressure depends on the time n energy density:

$$\begin{aligned} \varepsilon_{j-\frac{1}{2}}^{n+\frac{1}{2}} &= \varepsilon_{j-\frac{1}{2}}^{n-\frac{1}{2}} - \left(P + Q_R\right)_{j-\frac{1}{2}}^{n+\frac{1}{2}} \left[\frac{1}{\rho_{j-\frac{1}{2}}^{n+1}} - \frac{1}{\rho_{j-\frac{1}{2}}^{n}}\right] \\ &+ \frac{2a_{j-\frac{1}{2}}^{n+\frac{1}{2}} u_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{\rho_{j-\frac{1}{2}}^{n+\frac{1}{2}} R_{j-\frac{1}{2}}^{n+\frac{1}{2}}} \left(Q_{R\,j-\frac{1}{2}}^{\,n+\frac{1}{2}} - Q_{A\,j-\frac{1}{2}}^{\,n+\frac{1}{2}}\right) \\ &- \frac{4\pi}{a}\left[\left(a_j^{n+\frac{1}{2}}\right)^2 \left(R_j^{n+\frac{1}{2}}\right)^2 H_j^{n+\frac{1}{2}} - \left(a_{j-1}^{n+\frac{1}{2}}\right)^2 \left(R_{j-1}^{n+\frac{1}{2}}\right)^2 H_{j-1}^{n+\frac{1}{2}}\right] \end{aligned} \quad . \tag{47}$$

For an explicit formulation, an equation of state call is necessary to determine the intermediate value:

$$P_{j-\frac{1}{2}}^{n+\frac{1}{2}} = P\left(\frac{\rho_{j-\frac{1}{2}}^{n} + \rho_{j-\frac{1}{2}}^{n}}{2}, \varepsilon_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) \tag{48}$$

A second equation of state call after epsilon is updated produces the time n+1 value of pressure. For an implicit formulation, pressure and energy density are updated simultaneously:

$$P_{j-\frac{1}{2}}^{n+\frac{1}{2}} = \left(\frac{P_{j-\frac{1}{2}}^{n+1} + P_{j-\frac{1}{2}}^{n}}{2}\right),$$

with the time n+1 value of pressure obtained with an equation of state call using time n+1 values of density and energy density.

After the energy density is updated, the artificial heat flux can be calculated:

$$H_j^{n+1} = \begin{cases} 2h_0^2\left[\dfrac{\rho\Delta u_{j+\frac{1}{2}}^{n+\frac{1}{2}} \cdot \rho\Delta u_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{\rho\Delta u_{j+\frac{1}{2}}^{n+\frac{1}{2}} + \rho\Delta u_{j-\frac{1}{2}}^{n+\frac{1}{2}}}\right]\left(\varepsilon_{th\,j+\frac{1}{2}}^{\,n} - \varepsilon_{th\,j-\frac{1}{2}}^{\,n}\right) & \text{if } Q_{R\,j+\frac{1}{2}}^{\,n+\frac{1}{2}} > 0 \\ 0 & \text{if } Q_{R\,j+\frac{1}{2}}^{\,n+\frac{1}{2}} = 0 \end{cases} \quad . \tag{49}$$

The quantity C can now be obtained:

$$C_j^{n+1} = \frac{2.0}{R_j^{n+1}} \frac{R_{j+1}^{n+1} - R_{j-1}^{n+1}}{\Delta\mu_{j+\frac{1}{2}} + \Delta\mu_{j-\frac{1}{2}}} \left(Q_{R_j}{}^{n+1} - Q_{A_j}{}^{n+1} \right) + \frac{\rho_{j-\frac{1}{2}}^{n+1}}{4\pi a_j^{n+1} c^2} \left[\frac{H_j^{n+1}}{\left(\rho_j^{n+1}\right)^2 \left(R_j^{n+1}\right)^2} - \frac{H_j^n}{\left(\rho_j^n\right)^2 \left(R_j^n\right)^2} \right] . \tag{50}$$

The temporal metric coefficient is updated by a purely spatial integration inward from the edge:

$$f = \frac{\left[\dfrac{P_{j+\frac{1}{2}}^{n+1} + Q_{R_{j+\frac{1}{2}}}{}^{n+1} - P_{j-\frac{1}{2}}^{n+1} - Q_{R_{j-\frac{1}{2}}}{}^{n+1}}{\Delta\mu_j} + C_j^{n+1} \right]}{\rho_j^{n+1} w_j^{n+1}} , \tag{51}$$

$$a_{j-\frac{1}{2}}^{n+1} = a_{j+\frac{1}{2}}^{n+1} \exp(f) , \tag{52}$$

with

$$a_{J+\frac{1}{2}}^{n+1} = \left[1 - \frac{2Gm_J^{n+1}}{R_J^{n+1} c^2} \right] \left[1 + \frac{u_J^{n+1\,2}}{c^2} - \frac{2Gm_J^{n+1}}{R_J^{n+1} c^2} \right]^{-\frac{1}{2}} , \tag{53}$$

or

$$e^{\phi_{j-1}^{n+1}} = e^{\phi_j^{n+1}} \frac{\left[\dfrac{P_{j+\frac{1}{2}}^{n+1} + Q_{R_{j+\frac{1}{2}}}{}^{n+1} - P_{j-\frac{3}{2}}^{n+1} - Q_{R_{j-\frac{3}{2}}}{}^{n+1}}{2\Delta\mu_j} + C_j^{n+1} \right]}{\rho_j^{n+1} w_j^{n+1}} \tag{54}$$

with

$$e^{\phi_J^{n+1}} = \left[1 - \frac{2GM_J^{n+1}}{R_J^{n+1} c^2} \right] \left[1 + \frac{u_J^{n+1\,2}}{c^2} - \frac{2GM_J^{n+1}}{R_J^{n+1} c^2} \right]^{-\frac{1}{2}} \tag{55}$$

The proper volume coefficient is updated:

$$\Gamma_{j-\frac{1}{2}}^{n+1} = 4\pi \rho_{j-\frac{1}{2}}^{n+1} \left(R_{j-\frac{1}{2}}^{n+1} \right)^2 \Delta R_{j-\frac{1}{2}}^{n+1} \tag{56}$$

or

$$\Gamma_j^{n+1} = 1 + \frac{u_j^{n+1\,2}}{c^2} - \frac{2G_j^{n+1} m_j^{n+1}}{R_j^{n+1} c^2} \tag{57}$$

The mass is obtained by a spatial integration out from the center, where $m = 0$:

$$m_{j+1}^{n+1} = m_j^{n+1} + \Gamma_{j+1/2}^{n+1}\left(1 + \frac{\varepsilon_{j+1/2}^{n+1}}{c^2}\right)\Delta\mu_{j+1/2} + \frac{u_{j+1/2}^{n+1} H_{j+1/2}^{n+1}}{\rho_{j+1/2}^{n+1} c^4}\Delta\mu_{j+1/2} \tag{58}$$

5. SUMMARY

We derive the equations of spherical general relativistic hydrodynamics including a tensor artificial viscosity and an artificial heat conduction. Some general characteristics of artificial viscosities and an artificial heat conduction are discussed. A suggested difference formulation of the set of equations is given.

ACKNOWLEDGMENTS

I wish to thank Richard H. White for invaluable help, and F. Douglas Swesty, Doug Miller, Pat Crowley, and Grant Mathews for helpful discussions. I also thank the Fannie and John Hertz Foundation for a fellowship which partially supported this work. This work was also partially supported by US Department of Energy by the Lawrence Livermore National Laboratory under contract no. W-7405-ENG-48.

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Table I. Spherical General Relativistic Hydrodynamics Equations with Tensor Artificial Viscosity and Artificial Heat Flux

Velocity:

$$\dot{u} = -a\left[4\pi R^2\Gamma \frac{1}{\left(1+\frac{\varepsilon}{c^2}+\frac{P+Q_R}{\rho c^2}\right)}(P+Q_R)' + \frac{Gm}{R^2} + 4\pi(P+Q_R)R \right.$$
$$\left. + 4\pi R^2\Gamma \frac{1}{\left(1+\frac{\varepsilon}{c^2}+\frac{P+Q_R}{\rho c^2}\right)} C \right]$$

with $C = \frac{2\Gamma}{4\pi\rho R^3}(Q_R - Q_A) + \frac{\rho}{4\pi a c^2}\left[\frac{H}{\rho^2 R^2}\right]^{\cdot}$

Radius:

$$\dot{R} = au$$

Rest mass density:

$$\frac{(\rho R^2)^{\cdot}}{\rho R^2} = -\frac{au'}{R'} + \frac{4\pi G}{c^4}\frac{RH}{\Gamma} \quad \text{or } \rho = \Delta\mu \frac{\Gamma}{4\pi R^2 \Delta R}$$

Energy density:

$$\dot{\varepsilon} = -(P+Q_R)\left[\frac{1}{\rho}\right]^{\cdot} + \frac{2au}{\rho R}(Q_R - Q_A) - \frac{4\pi}{a}\left[a^2R^2H\right]'$$

Temporal metric coefficient:

$$\frac{a'}{a} = -\frac{(P+Q_R)'}{(\rho c^2 + \rho\varepsilon + P + Q_R)} - \frac{C}{(\rho c^2 + \rho\varepsilon + P + Q_R)} \quad \text{with}$$

$$a_{edge} = \left[1 - \frac{2GM}{Rc^2}\right]\left[1 + \frac{u^2}{c^2} - \frac{2GM}{Rc^2}\right]^{-1/2}$$

or

$$\phi' = -\frac{(P+Q_R)'}{(\rho c^2 + \rho\varepsilon + P + Q_R)} - \frac{C}{(\rho c^2 + \rho\varepsilon + P + Q_R)} \text{ with}$$

$$\phi_{edge} = \ln\left(\left[1 - \frac{2GM}{Rc^2}\right]\left[1 + \frac{u^2}{c^2} - \frac{2GM}{Rc^2}\right]^{-1/2}\right)$$

Mass:

$$m = \int_0^{\mu} 4\pi R^2 \rho(1+\varepsilon)\frac{dR}{d\mu} + \frac{uH}{\rho c^4}\, d\mu \quad \text{and} \quad \text{or} \quad m = \int_0^{\mu} \Gamma\rho(1+\varepsilon) + \frac{uH}{\rho c^4}\, d\mu$$

Proper volume coefficient:

$$\Gamma = 4\pi\rho R^2 R' \quad \text{or} \quad \Gamma = 1 + \frac{u^2}{c^2} - \frac{2G(m+I)}{Rc^2}$$

Enthalpy:

$$w = 1 + \frac{\varepsilon}{c^2} + \frac{P + Q_R}{c^2}$$